PHYS 161	**QUIZ 7**
April 10, 2009	**10 POINTS**

An automobile of mass 1200 kg rounds a curve at a speed of 25 m/s. The radius of the curve is 400 m and its banking angle is 6°.

In this solution, we will take $g = 10\ m/s^2$.

(i) [2] Draw a diagram to show all the forces. Write down the equations relating the forces and the speed of the car using Newton's second law.

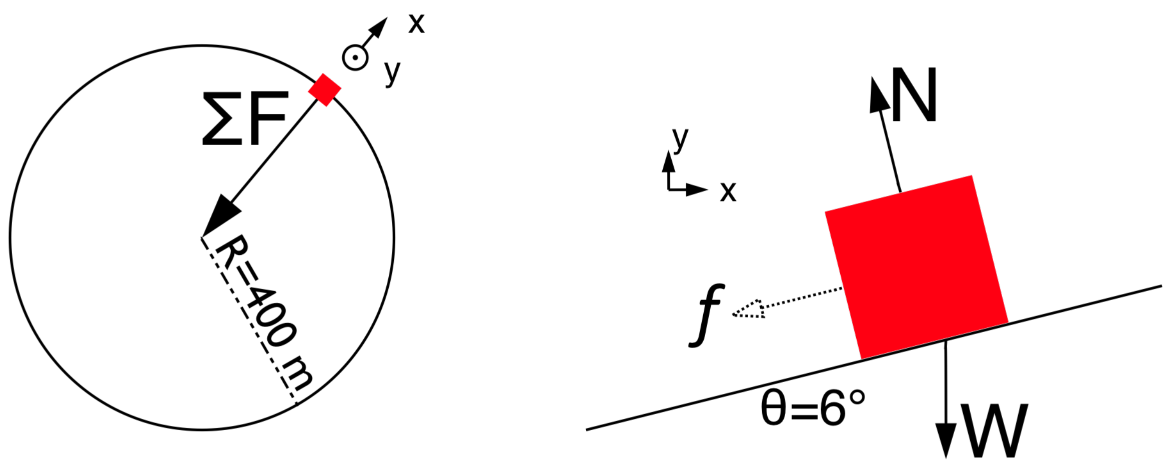

Figure 1: The track is shown on the left, the free-body diagram on the right. The car is red, as any hypothetical car ought be. The vector W is the weight (ie. mg) and the vector N is the normal force. The vector f is included for part (iii). In the left figure, y points out of the page and x points along the radius. In the right figure, y points up and x points right.

Newton's 2^{nd} law says that $\vec{F} = m\vec{a}$. Call the components $\hat{x}$ going to the right and $\hat{y}$ going up. We can break up each vector into x- and y-components:

$$\sum F_x = N_x + W_x \qquad \sum F_y = N_y + W_y \tag{1.1}$$

$$ma_x = -N\sin\theta \qquad ma_y = N\cos\theta - mg \tag{1.2}$$

Note the minus sign on the mg—the weight and the normal have y-components in the opposite direction.

(ii) [3] If the road is frictionless, can the automobile make the turn without slipping? Explain.

If the car does not slip, it must not accelerate up or down. Then, we know $a_y = 0$, so

$$N = \frac{mg}{\cos\theta} \tag{1.3}$$

Plugging that into the x-component equation, we find

$$ma_x = -\frac{mg}{\cos\theta}\sin\theta = mg\tan\theta \tag{1.4}$$

Thus, we find that

$$a_x = -g \tan\theta = -10\ m/s^2\ \tan 6^\circ \approx -1.05\ m/s^2 \tag{1.5}$$

We would like this car to undergo uniform circular motion with velocity $v = 25\ m/s$. So, we hope that the acceleration points towards the center of the circle. From the left diagram, we see that that is in the negative x direction. We hope that the a_x above matches the quantity that we get by requiring uniform circular motion:

$$a_x = -\frac{v^2}{R} = -\frac{(25\ m/s\ )^2}{400m} = -\frac{625m^2/s^2}{400m} \approx -1.56\ m/s^2 \tag{1.6}$$

Since the acceleration that we find by requiring $\vec{F} = m\vec{a}$ (-1.05 m/s^2) and the acceleration that we find by requiring uniform circular motion (-1.56 m/s^2) do not match, either $\vec{F} = m\vec{a}$ is incorrect, or the assumption that the car is undergoing uniform circular motion is incorrect. Since we trust Newton's laws, we conclude that the car cannot make the banked turn without slipping.

Note that this question may also be answered by comparing the forces given $F = ma$ and the force required for circular motion (mv^2/R inward). In this case, you will compare the numbers compared above, except with both multiplied by 1200 kg.

(iii) [1+2+2] Now assume that the road has friction. Write down Newton's second law for this case. What frictional force is needed for the auto to be able to turn without slipping? Which direction does the frictional force point when the car is turning?

If the road has friction, we need to account for that when we sum up the forces:

$$\sum F_x = N_x + W_x + f_x \qquad \sum F_y = N_y + W_y + f_y \tag{1.7}$$

$$ma_x = -N\sin\theta - f\cos\theta \qquad ma_y = N\cos\theta - mg - f\sin\theta \tag{1.8}$$

Now, to avoid slipping, we require $a_y = 0$. Let's also try to impose a requirement of circular motion–that is: let's try $a_x = -v^2/R$.

$$-m\frac{v^2}{R} = -N\sin\theta - f\cos\theta \qquad 0 = N\cos\theta - mg - f\sin\theta \tag{1.9}$$

We have two equations and two unknowns (N and f). You might say, "Hey, we know that $f = \mu N$!" Well, sure, but we don't know μ, so we still would have 2 equations and 2 unknowns. Since we're asked for the frictional force needed, it will be easier to just use f. Solving both equations for N we find:

$$N = \frac{1}{\sin\theta}\left(\frac{mv^2}{R} - f\cos\theta\right) \qquad N = \frac{1}{\cos\theta}(mg + f\sin\theta) \tag{1.10}$$

Setting these equal to each other, we find

$$\frac{1}{\sin\theta}\left(\frac{mv^2}{R} - f\cos\theta\right) = \frac{1}{\cos\theta}(mg + f\sin\theta) \tag{1.11}$$

Solving this equation, we find

$$f = \frac{mv^2\csc\theta - gmR\sec\theta}{R(\cot\theta + \tan\theta)} \approx \boxed{610N} \tag{1.12}$$

With $g = 9.8\ m/s^2$, f becomes $\approx \boxed{635N}$. You can examine how this answer behaves in various limits (ie. θ gets small, m gets large, v gets large, etc.) and everything makes sense. □