

Two springs of negligible mass, one with spring constant  $k_1$  and the other with spring constant  $k_2$ , are attached to the endstops of a level airtrack, as in the figure supplied. A glider attached to both the springs is located between them, as shown in the figure. When the glider is in equilibrium, the spring 1 is stretched by extension  $x_{i1}$  to the right of the unstretched length and spring 2 by  $x_{i2}$  to the left of the unstretched length. Now a horizontal force  $\vec{F}$  is applied to the glider to move it a distance  $x_a$  to the right from the equilibrium position. There is no friction between the glider and the airtrack.

- (i) [2] Show that  $x_{i2}$  is  $\frac{k_1}{k_2}x_{i1}$ .

Since initially the glider is in equilibrium, the net horizontal force must be zero. Since the airtrack is flat, this simply means:

$$F_1 + F_2 = 0 \tag{1.1}$$

$$-k_1x_{i1} + k_2x_{i2} = 0 \quad \Rightarrow \quad \boxed{x_{i2} = \frac{k_1}{k_2}x_{i1}} \tag{1.2}$$

- (ii) [3] Show that the work done on the spring 1 is  $\frac{1}{2}k_1(x_a^2 + 2x_ax_{i1})$ .

The work done is given by the integral of  $\vec{F} \cdot d\vec{x}$ . The force and displacement are aligned, so the dot product is equivalent to multiplying the magnitudes  $F dx$ . The force the spring resists with is  $-k_1x$ , so the force you must apply to the spring is  $k_1x$ . The force moves the glider from  $x = x_{i1}$  to  $x = x_a + x_{i1}$ , which are our limits of integration:

$$W = \int \vec{F} \cdot d\vec{x} \tag{1.3}$$

$$= \int_{x_{i1}}^{x_{i1}+x_a} (k_1x) dx \tag{1.4}$$

$$= \frac{1}{2}k_1x^2 \Big|_{x_{i1}}^{x_{i1}+x_a} = \frac{1}{2}k_1(x_{i1} + x_a)^2 - \frac{1}{2}k_1x_{i1}^2 = \boxed{\frac{1}{2}k_1(x_a^2 + 2x_ax_{i1})} \tag{1.5}$$

- (iii) [3] What is the work done on the spring 2?

The force will move the second spring closer to its equilibrium length, so the energy in the second spring will go down. The work done on the second spring is *decreasing* the energy in the spring,

and thus the work must be negative. So,

$$W = \int \vec{F} \cdot d\vec{x} \tag{1.6}$$

$$= \int_{x_{i2}}^{x_{i2}-x_a} k_2 x dx \tag{1.7}$$

$$= \frac{1}{2} k_2 x^2 \Big|_{x_{i2}}^{x_{i2}-x_a} = \frac{1}{2} k_2 (x_{i2} - x_a)^2 - \frac{1}{2} k_2 x_{i2}^2 \tag{1.8}$$

$$= \frac{1}{2} k_2 (x_a^2 - 2x_a x_{i2}) = \boxed{\frac{1}{2} k_2 \left( x_a^2 - 2x_a \frac{k_1}{k_2} x_{i1} \right)} \tag{1.9}$$

Looking at the first expression on the last line, we see that if  $x_a < x_{i2}$ , the work done is negative. That makes sense, because we are reducing the potential energy of the spring by unstretching it. If  $x_a = x_{i2}$ , the work is  $-\frac{1}{2} k_2 x_{i2}^2$ —in other words, when we totally unstretch the spring, the potential energy in that spring goes to zero! This is great! Also note that if we let  $x_a = 2x_{i2}$ , then we have compressed the spring past its equilibrium point by  $x_{i2}$ , and the work that we did on the spring is zero—this makes sense because a spring doesn't care if it is stretched or compressed: it only cares about the distance it is from equilibrium.

(iv) [2] What is the total work done by the force  $\vec{F}$ ?

The total work done by the force is just the sum of the work on each spring:

$$W = W_1 + W_2 \tag{1.10}$$

$$= \frac{1}{2} k_1 (x_a^2 + 2x_a x_{i1}) + \frac{1}{2} k_2 \left( x_a^2 - 2 \frac{k_1}{k_2} x_a x_{i1} \right) = \boxed{\frac{1}{2} (k_1 + k_2) x_a^2} \tag{1.11}$$

Which is a very nice result! The two springs conspire to behave as though they were just one spring with a constant that is the total of the two constants.

□