SOLUTION FOR THIRD MIDTERM
PHYSICS 161
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Administered Friday, May 4^{th}
50 Minutes Allotted.
Solutions by Evan Berkowitz.

50 POSSIBLE POINTS

In the figure provided, the pulley has a radius of 10 cm and a moment of inertia of 0.005 kg m^2 and spins on a frictionless axle. The mass of block B is 10 kg and that of block A is 5 kg. The coefficient of kinetic friction μ_k between the block A and the tabletop is 0.2. The blocks start to move from rest.

In the following solution, let m_a be the mass of block A, m_b be the mass of block B, I be the moment of inertia of the pulley, r be the radius of the pulley. T_1 is the tension between block A and the pulley, and T_2 is the tension between the pulley and block B.

(a) [7] Write down the equations describing the motion of the three moving parts in the problem (two blocks and the pulley). Using them find the linear acceleration of the blocks.
(b) [5] Find the tensions T_1 and T_2 in the strings.

Let us solve part the first two parts simultaneously:

We can see that block A will not have a net vertical force on it, so that the normal force N_a must equal $m_a g$. Thus, the magnitude of the kinetic friction f will be $\mu_k N_a = \mu_k m_a g$. Newton's second law for the blocks then say:

$$m_a a = T_1 - f \qquad\qquad m_b a = m_b g - T_2 \tag{1.1}$$

$$= T_1 - \mu_k m_a g \tag{1.2}$$

Note that we use the same acceleration a for both masses. This is true as long as the string does not stretch. Notice, then, that which direction is the positive direction and which is the negative direction matters significantly. If block A accelerates to the right, block B must accelerate down, to keep the string the same length. Thus, if we call rightward positive for block A, we should call downward positive for block B.

Those equations have three unknowns: a, T_1, and T_2. In a similar problem where we don't care about the motion of the pulley, we would find that the tensions T_1 and T_2 are equal, so that we would then have 2 equations with two unknowns. In this case, though, we require that the rope doesn't slide along the pulley. In other words:

$$\alpha = \frac{a}{r} \tag{1.3}$$

So that if block A accelerates to the right, the pulley accelerates clockwise. Now we can use the rotational version of Newton's second law:

$$I\alpha = \tau_{net} = rT_2 - rT_1 \tag{1.4}$$

Since we chose acceleration to the right as positive, we need to choose clockwise acceleration as positive too, which is why the torque from T_2 is positive and the torque from T_1 is negative. Notice that the forces from the tension are at right angles to their respective displacements from the center of the pulley, so that the $\sin\theta$ term in the torque becomes 1.

Summarizing, and substituting α as in (1.3), we need to solve:

$$m_a a = T_1 - \mu_k m_a g \qquad m_b a = m_b g - T_2 \qquad I\frac{a}{r} = r(T_2 - T_1) \tag{1.5}$$

Now we have *three* equations and three unknowns: a, T_1, and T_2. Solving, we find:

$$a = g\frac{(m_b - \mu_k m_a)}{\frac{I}{r^2} + m_a + m_b} \quad T_1 = g m_a\left(\mu + \frac{m_b - \mu_k m_a}{\frac{I}{r^2} + m_a + m_b}\right) \quad T_2 = g m_b\left(1 - \frac{m_b - \mu_k m_a}{\frac{I}{r^2} + m_a + m_b}\right) \tag{1.6}$$

$$a \approx \boxed{5.80\ m/s^2} \qquad T_1 \approx \boxed{39.03N} \qquad T_2 \approx \boxed{41.93N} \tag{1.7}$$

Notice that we wind up with the sensible result that the acceleration is smaller than the acceleration due to gravity, and that T_2 is more than T_1 (so that the pulley rotates clockwise).

(c) [2] If the block A was held fixed, keeping the whole system at rest, what would the tensions T_1 and T_2 have been?

If block A was held fixed, then we would be in static equilibrium. The easiest way to see what happens is by considering block B. If block B is in static equilibrium, the net force on it must be zero. Thus, T_2 must be equal to $m_b g$. Additionally, if the pulley is in static equilibrium, the net *torque* on it must be zero. Note that we don't know the force on the pulley supplied by the axle, so we cannot use that the net force on it should be zero. However, whatever force is supplied by the axle will be applied at the center of the pulley's rotation, and thus will not apply a torque (the r in $\tau = rF\sin\theta$ becomes zero). Since both T_1 and T_2 are at a distance r from the center and are both at right angles with respect to their lever arms, for the torques to cancel the forces must be equal in magnitude. Thus, T_1 must also be $m_b g$. That value is 100N (if $g = 10\ m/s^2$)

(d) [6] Use energy conservation to calculate the speed of the block B after it has descended a distance 0.5 m from rest.

What energies must we consider? First, there are the kinetic energies: the kinetic energy associated with the linear momentum of block A, the kinetic energy associated with the linear momentum of block B, and the kinetic energy associated with the rotational motion of the pulley. Then we must consider the potential energies. Let us choose the convention where block B starts with zero potential energy so that the potential energy becomes negative as block B falls. We need not consider the potential energies of the pulley or of block A because those potential energies will not change, and so will cancel out on both sides in our energy conservation equation.

Finally, we need to consider the energy lost to friction. Since the frictional force does not depend on position, the energy lost, $\int F dx$ is simply $F\Delta x$, and $F = \mu_k N = \mu_k m_a g$.

At the beginning, all the velocities are zero, and the displacement is zero so that the friction hasn't sapped any energy yet. So, we have that the initial energy is zero. The final energy is just the sum of the kinetic and potential energies, accounting for the energy lost to friction:

$$0 = E_i = E_f = \frac{1}{2}m_a v^2 + \frac{1}{2}m_b v^2 + \frac{1}{2}I\omega^2 - m_b g h + \mu_k m_a g h \tag{1.8}$$

where h is the height that block B has fallen (and the distance that block A has slid on the table) and ω is the rotational velocity of the pulley. Note that we use the same v for the kinetic energies of both block A and block B—this is the condition that the string does not stretch. A similar notion gives us that $\omega = v/r$. So, we find:

$$(m_b - \mu_k m_a)gh = \frac{1}{2}\left(m_a + m_b + \frac{I}{r^2}\right)v^2 \tag{1.9}$$

Solving for v we find,

$$\boxed{v^2 = 2gh\frac{m_b - \mu_k m_a}{m_a + m_b + \frac{I}{r^2}}} \tag{1.10}$$

If there is no friction, and the pulley has no mass, this reduces to

$$v^2 = 2gh\frac{m_b}{m_a + m_b} \tag{1.11}$$

which you should be able to show is the correct answer if there is indeed no friction and we ignore the pulley.

The numerical result of (1.10) (with $h = 0.5m$),is:

$$v^2 \approx 5.80m^2/s^2 \qquad \Rightarrow \qquad \boxed{v \approx 2.41\ m/s} \tag{1.12}$$

□

PHYS 161	**Exam 2 - QUESTION 2**
May 9, 2009	**16 POINTS**

Consider a frictionless track ABC as in the figure provided. A block of mass m_1=5.00 kg is released from A. It makes a head-on elastic collision with a block of mass m_2=10.0 kg at the point B.

Let h be the height of the first mass above the flat track.
(a) [4] What are the speeds of the two blocks after collision?

We are told that the collision is elastic, ie. that energy is conserved in the collision. Momentum is always conserved in collisions. So, we just need to know the energy before the system has before collision and how much momentum the system has before the collision. These we can find by conservation of energy, because we are told the starting height of m_1. At the beginning of this process, all the energy is potential energy, and is equal to $m_1 g h_a$, where h_a is the height at point A (ie. 5m).

Let the velocities of m_1 and m_2 be v and V, with before and after carrying subscripts of b and f. Since m_2 starts at rest, $V_b = 0$. Conserving energy through the fall of m_1, we know that

$$\frac{1}{2}m_1 v_b^2 = E_b = E_a = m_1 g h_a \tag{2.1}$$

where E_a is the energy when m_1 is at A, by conservation of energy. This means that

$$v_b = \sqrt{2gh_a} \tag{2.2}$$

Conserving momentum before and after the collision,

$$m_1 v_b = m_1 v_f + m_2 V_f. \tag{2.3}$$

Conserving energy before and after the collision,

$$\frac{1}{2}m_1 v_b^2 = E_b = E_f = \frac{1}{2}m_1 v_f^2 + \frac{1}{2}m_2 V_f^2 \tag{2.4}$$

In the preceding two equation we have two unknowns (v_f and V_f). Solving these two equations, we find

$$v_f = \frac{m_1 - m_2}{m_1 + m_2} v_b \qquad\qquad V_f = \frac{2m_1}{m_1 + m_2} vb \tag{2.5}$$

Plugging in our result from conserving energy, (2.2), in these quantities, we find

$$v_f = \frac{m_1 - m_2}{m_1 + m_2}\sqrt{2gh_a} \qquad\qquad V_f = \frac{2m_1}{m_1 + m_2}\sqrt{2gh_a} \tag{2.6}$$

Plugging in numbers, ($g = 10\ m/s^2$) we find:

$$\begin{aligned} v_f &= \frac{5kg - 10kg}{5kg + 10kg}\sqrt{2 \times 10\ m/s^2\ \times 5m} & V_f &= \frac{2 \times 5kg}{5kg + 10kg}\sqrt{2 \times 10\ m/s^2\ \times 5m} \\ &= -\frac{1}{3} \times 10\ m/s\ = \boxed{-3.33\ m/s} & &= +\frac{2}{3} \times 10\ m/s\ = \boxed{+6.66\ m/s} \end{aligned} \tag{2.7}$$

In later parts we'll use $-10/3\ m/s$ and $20/3\ m/s$ for our values of v_f and V_f.

(b) [4] How far up the slope does the block of mass m_1 climb after the collision?

Again we can conserve energy. The energy in mass m_1 immediately following the collision is determined by v_f. When the mass reaches its peak at a height h_{max}, its velocity and thus its kinetic energy will be zero. So, conserving energy tells us:

$$\frac{1}{2}m_1 v_f^2 = m_1 g h_{max} \qquad \Rightarrow \qquad h_{max} = \frac{v_f^2}{2g} \tag{2.8}$$

Plugging in our quantities ($g = 10\ m/s^2$) gives

$$h_{max} = \frac{(-10/3\ m/s\ )^2}{2 \times 10\ m/s^2} = \frac{5}{9}m = \boxed{.55m} \tag{2.9}$$

(c) [4] If the horizontal surface in the figure had friction with $\mu_k = 0.2$, how far would the second block go after the collision?

The force of kinetic friction is always $\mu_k N$, which in this case is $\mu_k(m_2 g)$. The force doesn't depend on position, so to find the work done on the second block we can evaluate:

$$W = \int F dx = F\Delta x = -\mu_k m_2 g \Delta x \tag{2.10}$$

Note the minus sign—this comes in because friction always opposes the direction of the motion.

The energy that m_2 starts with is solely kinetic, and thus is $\frac{1}{2}m_2 V_f^2$. If it comes to rest, its energy will be 0, as the whole surface is at height=0. Conserving energy, we find

$$E_{rest} = E_f + W \qquad \Rightarrow \qquad 0 = \frac{1}{2}m_2 V_f^2 + (-\mu_k m_2 g \Delta x)\,. \tag{2.11}$$

Solving this equation for Δx, we find,

$$\Delta x = \frac{V_f^2}{2\mu_k g} \tag{2.12}$$

Note that if there is no friction ($\mu_k = 0$), then the expression goes to infinity—in other words, the block never stops. That is the correct behavior when there is no friction!

Plugging in our numbers, we find

$$\Delta x = \frac{(20/3\ m/s\ )^2}{2 \times .2 \times 10\ m/s^2} = \frac{100}{9}m \approx \boxed{11.1m} \tag{2.13}$$

(d) [4] If the two blocks got stuck together after the collision, what would be the final velocity of the entangled system?

If the two blocks get stuck together, the collision is not elastic, so we may only use conservation of momentum at the instant of collision. If they get stuck together then they act as one object of mass $m_1 + m_2$. Conserving momentum we find,

$$m_1 v_b = (m_1 + m_2) v_s \tag{2.14}$$

Where v_s is the velocity when they get stuck together. Solving, we see

$$v_s = \frac{m_1}{m_1 + m_2} vb = \frac{m_1}{m_1 + m_2} \sqrt{2gh_a} \tag{2.15}$$

using our result (2.2) which came from conserving energy as block m_1 fell—this conservation of energy is still allowed, because it is just the conversion of potential energy to kinetic energy, and does not have to do with the instant of the collision. So, we find

$$v_s = \frac{1}{3} \times 10\ m/s\ = \boxed{3.33\ m/s} \tag{2.16}$$

□

(a) [2] Two cylinders of the same mass and same dimensions are set into rotation about their long axes with the same angular speed. One is hollow and the other is solid. The same torque is applied to both the rotating cylinders. The direction of the torque is opposite that of the rotational motion. Which cylinder will stop first? Explain.

The hollow cylinder has a larger moment of inertia (MR^2) than the solid cylinder ($\frac{1}{2}MR^2$) Since the torque is equal to $I\alpha$ (the moment of inertia times the angular acceleration), if the two experience the same torque, the solid cylinder will have an α that is twice as large, and thus will spin down more quickly. So, $\boxed{\text{the solid cylinder will stop first}}$. This is why yo-yos typically have their mass located around the rim and not evenly distributed throughout their halves—placing the mass around the rim decreases the effect of the friction between the yoyo's axle and the string.

(b) [2] A block sliding along a frictionless surface with a speed v collides with and compresses the spring. The maximum compression is 1.4 cm. If the block collied with a velocity $2v$ with the same spring, the spring's maximum compression would be:

(i) 0.35 cm (ii) 0.7 cm (iii) 1.0 cm (iv) 1.4 cm (v) 2.0 cm (vi) 2.8 cm (vii) 5.6 cm.

The energy of the block before touching the spring is $\frac{1}{2}mv^2$ and at maximum compression, that must equal $\frac{1}{2}kx^2$. Equating,

$$\frac{1}{2}mv^2 = \frac{1}{2}kx_{max}^2 \qquad \Rightarrow \qquad \frac{v}{x_{max}} = \sqrt{\frac{k}{m}} = \text{constant.} \tag{3.1}$$

So, if we double the incoming velocity, we must double the maximum compression in order to conserve energy. Thus, the maximum compression is $2 \times 1.4\text{cm} = \boxed{(vi)\ 2.8cm}$.

(c) [4] The figure below gives the potential energy function of a particle.

• (i) Rank the regions AB, BC, CD, DE according to the magnitude of the force on the particle, greatest first.

The force is equal to the negative derivative of the potential ($F = -\frac{dV}{dx}$), so the force will have the largest magnitude where the magnitude of the slope of the potential is greatest. The sorting the magnitude of the slopes from greatest to least gives: AB, CD, with BC and DE tied, because they are flat and thus exert 0 force.

• What is the values of E_{max} if the particle is to be:

• (ii) trapped in the potential well on the left,

If the particle is to be trapped in the left, then it must have turning points (where total energy = potential energy) on both sides of the left well. The largest possible energy where it can have one on each side is energy 5. If it has more than energy 5, it can escape to the right, across point D.

• (iii) trapped in the potential well on the right,

If the particle is to be trapped in the right, then it must have turning points on both sides of the right well. The largest possible energy where it can have one on each side is energy 5, because you if it has more than energy 5, it can escape to the left, across point E.

• (iv) able to move between the points A and H but not move out of this region?

If the particle is to be trapped between A and H, there must be turning points there. The largest possible energy where it can have a turning point at H is energy 6, because if it has more energy it can escape by going to the right. To escape to the left, it would need an energy equal to energy 8.

• For the situation in (iv), in which region (BC, DE, FG) will the particle have the greatest and least speed.

The particle will have the greatest speed in region FG. Since the total energy is conserved, the kinetic energy (and thus the velocity) will be greatest when the potential energy is least.

(d) [2] In a supernova explosion, a typical star mass remains roughly the same but its radius decreases by a large factor. Does its angular speed decrease, increase, or remain the same?

Just like an ice skater pulling his or her arms inward, $\boxed{\text{the supernova's angular speed will increase}}$. This is because the angular momentum is conserved, but the moment of inertia is going down, so the angular speed must go up to keep the angular momentum constant.

(e) [2] Why cant you put your heels firmly against a wall and bend over without falling?

If you bend over with your heels against a wall, the force of gravity pulls your torso downward. With respect to your heels, this creates a torque. The only way to not fall over would be to have a torque trying to rotate you in the opposite sense. There are two ways to do this: usually, when you are free standing, you put your behind in the opposite direction from the way you are bending over. Then, gravity exerts a torque there in the opposite sense. Alternatively, you could try to use your feet and toes as a lever, but your feet are so short that the force that they would have to supply to balance the torque from your torso is very large. Since you cannot balance the torque, you get an angular acceleration, ie. fall over.

(f) [2] A mouse is initially at rest on a turntable mounted on a frictionless vertical axis. The mouse begins to walk clockwise around the perimeter of the table. What happens to the table? If instead the mouse walked towards the center of the table along a radius, what would happen? Explain.

Since the system doesn't have external torques on it, angular momentum must be conserved—angular momentum must remain 0 (everything is at rest). Thus, if the mouse walks clockwise he will get a negative angular momentum. To keep angular momentum at 0, the turntable must pick up a positive angular momentum, and thus will spin counterclockwise.

If the mouse walks toward the center along a radius instead, nothing would happen—the initial angular momentum would be 0, and since the mouse doesn't get an angular momentum around the axle if he walks along the radius, the turntable need will not spin, so that the total angular momentum remains 0.

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